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BCS-BEC Crossover and Chiral Anomaly in p -Wave Superfluids with the Symmetry of A1-Phase

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Abstract We solve the Leggett equations for BCS-BEC crossover in the 3D resonance p -wave superfluid with the symmetry of A1-phase. We calculate sound velocity, normal density and specific heat for the BCS-domain ($\mu > 0$), BEC-domain ($\mu < 0$) as well as close to important point $\mu = 0$ in 100% polarized case. We find the indications of quantum phase-transition close to the point $\mu(T = 0) = 0$. Deep in BCS and BEC-domains the crossover ideas of Leggett and Nozieres, Schmitt-Rink work pretty well.

We discuss the spectrum of orbital waves, the paradox of intrinsic angular momentum and complicated problem of chiral anomaly in BCS A1-phase at $T = 0$. We present two different approaches to a chiral anomaly: one based on supersymmetric hydrodynamics, another one on the formal analogy with Dirac equation in quantum electrodynamics (QED theory). We evaluate the damping of nodal fermions due to different decay processes in superclean case at $T = 0$ and find that we are in a ballistic regime $\omega\tau \gg 1$. We propose to use aerogel or nonmagnetic impurities to reach hydrodynamic regime $\omega\tau \ll 1$ at $T = 0$.

We discuss the concept of spectral flow and exact cancellations between time-derivatives of anomalous and quasiparticle currents in the equation for the conservation of total linear momentum. We propose to derive and solve a kinetic equation for nodal quasiparticles both in the hydrodynamic and in the ballistic regimes to demonstrate this cancellation explicitly. We briefly discuss the role of the other residual

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interactions different from damping and invite experimentalists to measure the spectrum and damping of orbital waves in A-phase of ^3He at low temperatures.

Keywords BCS-BEC crossover · Feshbach resonance · Superfluid ^3He -A · Chiral anomaly · Spectral flow

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1 Introduction

The first experimental results on p -wave Feshbach resonance [1–6] in ultracold fermionic gasses ^{40}K and ^6Li make the field of quantum gasses closer to the interesting physics of superfluid ^3He and the physics of unconventional superconductors such as Sr_2RuO_4 . In this context it is important to build the bridge between the physics of ultracold gasses and the low-temperature physics of quantum liquids and anomalous superconductors and thus to enrich both communities with the experience and knowledge accumulated in each of these fields. The purpose of the present paper is first of all to describe the transition from the weakly bound cooper pairs with p -wave symmetry to strongly bound local p -wave pairs (molecules) and try to reveal the nontrivial topological effects connected with the presence of the nodes in the superfluid gap of the 100%-polarized p -wave A1-phase in 3D. Note, that the A1-phase symmetry is relevant both to ultracold Fermi-gasses in the regime of p -wave Feshbach resonance and to superfluid ^3He -A in the presence of large magnetic field or large spin-polarization. We pay the special attention to the spectrum of collective excitations and to the superfluid hydrodynamics of the A1-phase at $T = 0$, where the topological effects are very pronounced, especially in the BCS-domain. We propose the experimental verification of the different approaches connected with the complicated problem of chiral anomaly and mass-current nonconservation in superfluid A1-phase of ^3He in superclean case and in the presence of aerogel as well as for the A1 p -wave condensates in magnetic traps in the presence of Josephson tunneling currents.

Our paper is organized as follows. Section 1 provides an Introduction. In Sect. 2 we briefly comment on the recent experiments on p -wave Feshbach resonance and describe the global phase-diagram for 100%-polarized p -wave resonance superfluids in 3D. In Sect. 3 we describe the quasiparticle spectrum and nodal points in A1-phase. In Sect. 4 we solve mean-field Leggett equations for triplet superfluids with the symmetry of A1-phase at $T = 0$ and study the behavior of superfluid gap Δ , chemical potential μ , and sound velocity c_s deep in BCS- ($\mu > 0$) and BEC-domains ($\mu < 0$) as well as close to the interesting point $\mu = 0$. In Sect. 5 we study the temperature behavior of the normal density ρ_n and specific heat C_v in BCS-domain, in BEC-domain, and close to $\mu = 0$, where we find the indications of quantum phase-transition.

In Sect. 6 we describe the orbital waves spectrum in BCS- and BEC-domains of the A1-phase and describe the complicated problem of chiral anomaly (mass-current nonconservation) which exists in the superfluid hydrodynamics of A1-phase in BCS-domain at $T \rightarrow 0$. In Sects. 7 and 8 we present two different approaches to the calculation of anomalous current: first one based on supersymmetric hydrodynamics [7]

and the second one—on the analogy with Dirac equation in quantum electrodynamics (QED) [8–12]. Note that both approaches are very general. The first one is based on the inclusion of fermionic Goldstone mode in low-frequency hydrodynamic action [7]. It can be useful for all nodal superfluids and superconductors with zeroes of the superconductive gap such as $^3\text{He-A}$, Sr_2RuO_4 , UPt_3 , UNi_2Al_3 , $\text{U}_{1-x}\text{Th}_x\text{Be}_{13}$ and so on [13]. The second approach is also very nice and general. It is connected with the appearance of the Dirac-like spectrum of fermions with zero mode [8–12] which also arises in many condensed-matter systems such as $^3\text{He-A}$, chiral superconductor Sr_2RuO_4 , organic conductor $\alpha\text{-(BEDT-TTF)}_2\text{I}_3$, 2D semiconductors, or recently discovered graphene [13–17]. In Sect. 9 we evaluate the damping in superclean A1-phase at $T = 0$ due to different decay processes and come to the conclusion that we are in a ballistic regime $\omega\tau \gg 1$ at $T = 0$. We propose to use aerogel or non-magnetic impurities to reach hydrodynamic regime $\omega\tau \ll 1$. We discuss the concept of spectral flow and exact cancellations of anomalies between time-derivatives of anomalous and quasiparticle currents in the equation of the conservation of total linear momentum. We also propose to derive both in a hydrodynamic and in a ballistic regimes a kinetic equation for nodal quasiparticles and to demonstrate this cancellation explicitly. In Sect. 10 we provide our conclusions and acknowledgements. We also invite experimentalists to measure the spectrum and damping of orbital waves in $^3\text{He-A}$ phase at low temperatures $T \ll T_C$ and thus to help to resolve the paradox of the orbital momentum. We also propose to extend the measurements of orbital inertia and orbital viscosity in non-singular vortex textures in A-phase [18, 19] to low temperatures with the help of the creation of spin-polarization. Finally we propose to measure a Josephson current between two 2D films of axial and planar phases with an attempt to extract directly the difference between topological charges $\Delta Q = 1$ in these phases.

2 Feshbach Resonance and Phase-Diagram for 100%-Polarized p -Wave Resonance Superfluids

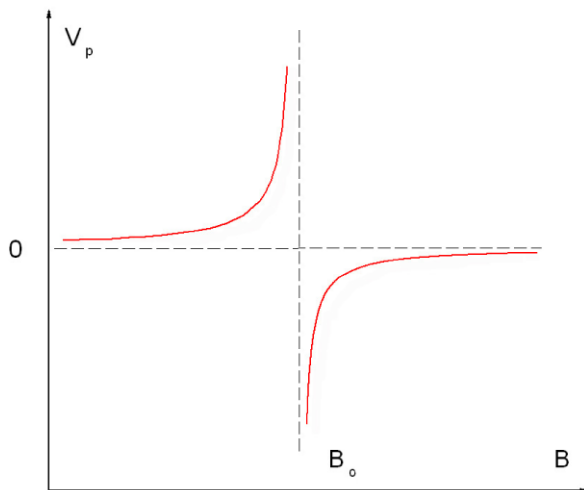
In the first experiments on p -wave Feshbach resonance the experimentalists measure the molecule formation in the ultracold fermionic gas of ^6Li -atoms close to resonance magnetic field B_0 [1–5].

In the last years the analogous experiments on p -wave molecules formation in spin-polarized fermionic gas of ^{40}K -atoms were started [6]. The lifetime of p -wave molecules is rather short yet [1–6]. However the physicists working in ultracold gasses have started intensively to study the huge bulk of experimental and theoretical wisdom accumulated in the physics of superfluid ^3He and anomalous complex superconductors (see [20, 21]).

To understand the essence of p -wave Feshbach resonance we should recollect the basic formula on p -wave scattering amplitude in vacuum from [22–25]:

$$f_{l=1}(E) = \frac{pp'}{\frac{1}{V_p} + \frac{2mE}{\pi r_0} + i(2mE)^{3/2}}, \quad (1)$$

Fig. 1 (Color online) The sketch of the p -wave Feshbach resonance. For $B = B_0$ the scattering volume V_p diverges



where $l = 1$ is an orbital momentum in the p -wave channel, E is a two particle energy, $V_p = r_0^2 a_p$ is scattering volume, a_p is p -wave scattering length, r_0 is the range of the interaction, p, p' are incoming and outgoing momenta. For Feshbach resonance in fermionic systems $p \sim p' \sim p_F$ and usually $p_F r_0 < 1$. The scattering length a_p and hence the scattering volume V_p is divergent in the resonance magnetic field B_0 (see Fig. 1) $1/V_p = 1/a_p = 0$. The imaginary part of the scattering amplitude f_p is small and nonzero only for positive energies $E > 0$, so p -wave Feshbach resonance is intrinsically narrow. Note that for negative energies $E < 0$ there is a molecular bound state:

$$|E_b| = \frac{\pi r_0}{2m V_p} = \frac{\pi}{2m r_0 a_p}.$$

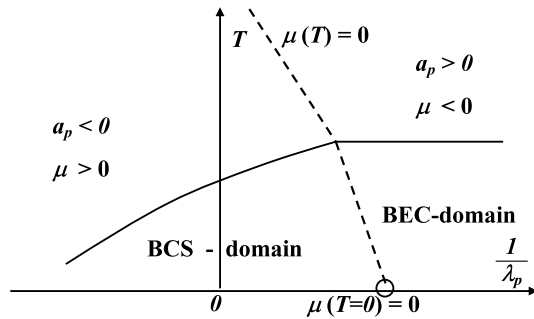
In the unitary limit the molecular binding energy $|E_b| \rightarrow 0$.

The first theoretical articles on p -wave Feshbach resonance often deal with mean-field two-channel description of the resonance [23–25]. In our paper we will study p -wave Feshbach resonance in the framework of one-channel description, which is more close to the physics of superfluid ^3He and captures rather well the essential physics of BCS-BEC crossover in p -wave superfluids.

In magnetic traps (in the absence of the so-called dipolar splitting) people usually study fully (100%) polarized gas or more precisely—one hyperfine component of the gas. On the language of ^3He they study the pairs with $S_{tot} = S_z^{tot} = 1$, or $|\uparrow\uparrow\rangle$ -pairs. In our paper we will consider p -wave triplet A1-phase in 3D where just $S_{tot} = S_z^{tot} = 1$.

The qualitative picture of the global phase-diagram of the BCS-BEC crossover in 100%-polarized A1-phase is presented in Fig. 2. In gross features it resembles the phase-diagram of the BCS-BEC crossover for s -wave pairing (see [26–29] for more details). However there is a very interesting question about the origin of the point $\mu(T = 0) = 0$ for 3D A1-phase. We will show that in the point $\mu(T = 0) = 0$ we probably deal with a quantum phase-transition [30, 31].

Fig. 2 Qualitative picture of the BCS-BEC crossover in 100%-polarized A1-phase in the axis T (temperature) and $1/\lambda_p$ —inverse gas parameter for p -wave superfluids ($\lambda_p = V_p p_F^3$, V_p is scattering volume), μ is chemical potential, a_p is scattering length ($V_p = r_0^2 a_p$). We indicate the line where $\mu(T) = 0$ and a quantum phase-transition point $\mu(T=0) = 0$



BCS-domain where chemical potential $\mu > 0$, occupies on the global phase-diagram the region of negative values of the gas parameter $\lambda_p = V_p p_F^3 < 0$ (or the negative values of the scattering length a_p). It stretches also to the small positive values of the inverse gas parameter $1/\lambda_p \leq 1$ and is separated from the BEC-domain (where $\mu < 0$ and the inverse gas parameter is large and positive $1/\lambda_p \geq 1$) by the line $\mu(T) = 0$. Usually in the regime of Feshbach resonance the density of “up” spins $n = p_F^3/6\pi^2$ is fixed. Deep inside BCS-domain (for small absolute values of the gas parameter $|\lambda_p| \ll 1$) we have the standard BCS-like formula for the critical temperature of the A1-phase:

$$T_{Cp} = 0.1\varepsilon_F e^{-\frac{\pi}{2|\lambda_p|}}, \quad (2)$$

where the preexponential factor for the 100% polarized A1-phase is defined by second order diagrams of Gor’kov and Melik-Barchudarov type [32] and approximately equals to $0.1\varepsilon_F$ [33–36].

Deep in BEC-domain ($\lambda_p \ll 1$) the well-known formula of Einstein is working in principal approximation for Bose-condensation of p -wave molecules with the density $n/2$ and the mass $2m$:

$$T_{Cp} = 3.31 \frac{(n/2)^{2/3}}{2m}. \quad (3)$$

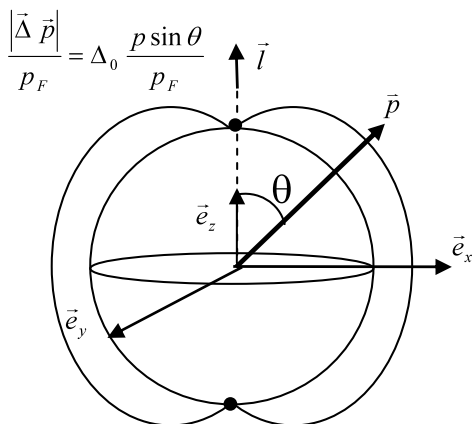
In the unitary limit $1/\lambda_p = 0$. Hence here $T_{Cp} \approx 0.1\varepsilon_F$ and we are still in BCS-regime (see [26–29]). In the rest of the paper we will consider low temperatures $T \ll T_C$, so we will work deep in the superfluid parts of BCS and BEC-domains of the A1-phase.

3 Quasiparticle Energy and Nodal Points in A1-Phase

For standard s -wave pairing the quasiparticle spectrum reads:

$$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \Delta_0^2}. \quad (4)$$

Fig. 3 The topology of the superfluid gap in A1-phase. θ is the angle between momentum \vec{p} and the axis of orbital momentum quantization $\vec{l} = \vec{e}_z$. There are two nodes in the quasiparticle spectrum corresponding to the south and north poles



It has no zeroes (no nodes), so the topology of the s -wave pairing problem is trivial. For triplet A1-phase however:

$$E_p = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \frac{|\vec{\Delta}\vec{p}|^2}{p_F^2}}, \quad (5)$$

where $\vec{\Delta} = \Delta_0(\vec{e}_x + i\vec{e}_y)$ —is the complex order parameter in A1-phase, Δ_0 is the magnitude of the superfluid gap. In fact: $|\vec{\Delta}\vec{p}|^2 = \Delta_0^2 p^2 \sin^2 \theta = \Delta_0^2 [\vec{p}\vec{l}]^2$, where $\vec{l} = [\vec{e}_x, \vec{e}_y]$ —is the unit vector of orbital momentum (see Fig. 3). Note that p_F is fixed by fixed density n . Angle θ is the angle between momentum \vec{p} and the orbital momentum quantization axis $\vec{l} = \vec{e}_z$.

For $\mu > 0$ (BCS-domain) there are two nodes in the spectrum for $p^2/2m = \mu$ and for $\theta = 0$ or π . For $\mu < 0$ (BEC-domain) there are no nodes. The important point $\mu = 0$ is a boundary between the totally gapped BEC-domain and the BCS-domain with two nodes of the quasiparticle spectrum corresponding to the south and north poles of Fig. 3. Often this point for $T = 0$ is called the point of *topological* quantum phase-transition [37–41].

4 Leggett Equations for A1-Phase

The Leggett equations for the 100%-polarized A1-phase in 3D are the evident generalization of the standard Leggett equations for the s -wave BCS-BEC crossover [26–29, 42]. The first equation reads:

$$n = \frac{p_F^3}{6\pi^2} = \int_0^{1/r_0} \frac{p^2 dp}{2\pi^2} \int_{-1}^1 \frac{dx}{2} \left(1 - \frac{\xi_p}{E_p}\right) \frac{1}{2}, \quad (6)$$

where $\xi_p = (\frac{p^2}{2m} - \mu)$, $E_p = \sqrt{\xi_p^2 + \frac{\Delta_0^2 p^2}{p_F^2} \sin^2 \theta}$ —is a quasiparticle spectrum, $x = \cos \theta$. This equation defines the chemical potential μ for fixed density n .

The second self-consistency equation defines the magnitude of the superfluid gap Δ_0 . It reads:

$$-\pi m \operatorname{Re} \frac{1}{f_{l=1}(2\mu)} = \int_{-1}^1 \frac{dx}{2} \int_0^{1/r_0} p^4 dp \left\{ \frac{1}{E_p} - \frac{1}{\xi_p} \right\}, \quad (7)$$

where $\operatorname{Re} \frac{1}{f_{l=1}(2\mu)} = (\frac{1}{V_p} + \frac{4m\mu}{\pi r_0})$ —is a real part of an inverse scattering amplitude in p -wave channel for total energy $E = 2\mu$ of colliding particles. This energy is relevant for pairing problem, so $f_{l=1}(E)$ should be replaced by $f_{l=1}(2\mu)$ in Leggett equations. In (7) $V_p = r_0^2 a_p$.

The solution of Leggett equations yields deep in the BCS-domain:

$$\Delta_0 \sim \varepsilon_F e^{-\frac{\pi}{2|\lambda_p|}} \sim T_{Cp}; \quad \mu \approx \varepsilon_F > 0. \quad (8)$$

The sound velocity reads in 3D:

$$c_S = \left(\frac{n}{m} \frac{d\mu}{dn} \right)^{1/2} = \frac{v_F}{\sqrt{3}}. \quad (9)$$

For $1/|\lambda_p| = 0$: $\Delta_0 \sim \varepsilon_F$ and hence unitary limit is still inside BCS-domain.

Deep in BEC-domain:

$$\Delta_0 \approx 2\varepsilon_F \sqrt{p_F r_0} \ll \varepsilon_F \quad \text{for } p_F r_0 \ll 1, \quad (10)$$

and chemical potential $\mu = -\frac{|E_b|}{2} + \frac{\mu_B}{2} < 0$, where as we already mentioned:

$$|E_b| = \frac{\pi}{2mr_0 a_p} \quad (11)$$

is a binding energy of a triplet pair (molecule).

Accordingly:

$$\mu_B \approx \frac{4\varepsilon_F}{3} \sqrt{p_F r_0} \quad (12)$$

is a bosonic chemical potential which governs the repulsive interaction between two p -wave molecules [26–29].

The sound velocity deep in BEC-domain reads:

$$c_S = \left(\frac{n_B}{2m} \frac{d\mu_B}{dn_B} \right)^{1/2} \approx \frac{v_F}{\sqrt{3}} \sqrt{p_F r_0} \ll v_F \quad \text{for } p_F r_0 \ll 1, \quad (13)$$

where $n_B = n/2$ is bosonic density.

Close to the interesting point $\mu \rightarrow 0$ (more rigorously $|\mu| < \Delta_0^2/\varepsilon_F$) we have:

$$\Delta_0(\mu = 0) = 2\varepsilon_F \sqrt{p_F r_0} \quad (14)$$

for the magnitude of the superfluid gap.

For the gas parameter λ_p in the point $\mu = 0$ we have:

$$\lambda_p(\mu = 0) = \frac{3\pi}{4} > 0. \quad (15)$$

Hence the interesting point $\mu = 0$ is effectively in BEC-domain (in the domain of positive p -wave scattering length $a_p > 0$). Accordingly for $\mu = 0$ the binding energy:

$$|E_b| = \frac{4}{3}\varepsilon_F(p_F r_0). \quad (16)$$

The sound velocity squared for $\mu = 0$ reads:

$$c_s^2 = \frac{v_F^2}{3}(p_F r_0) \quad (17)$$

and coincides with the expression (10) obtained deep in BEC-domain. The careful analysis of Leggett equations close to the point $\mu = 0$ shows that the derivative $\frac{\partial \Delta}{\partial \mu}$ also have no singularities at this point. The second derivative $\frac{\partial^2 n}{\partial \mu^2}$ is also continuous at $\mu = 0$, so the anomaly will appear only in higher derivatives in qualitative agreement with numerical calculations of [43, 44] in 3D.

At the same time the careful analysis of compressibility in 2D case [45–47] shows continuous behavior but with a kink already in $\frac{\partial n}{\partial \mu}$ in 100%-polarized ($p_x + ip_y$) phase for $\mu = 0$ on the level of analytical calculation as well as in numerics [43, 44].

To be more specific:

$$\frac{\partial n}{\partial \mu} \propto 1 + \frac{\mu \varepsilon_F}{\Delta_0^2} [1 - \text{sign } \mu] \quad (18)$$

and thus $\frac{\partial n}{\partial \mu} \propto 1$ for $\mu \rightarrow +0$, and $\frac{\partial n}{\partial \mu} \propto 1 + \frac{2\mu \varepsilon_F}{\Delta_0^2}$ for $\mu \rightarrow -0$.

5 Specific Heat and Normal Density at Low Temperatures $T \ll T_C$

In this section we study the thermodynamic functions namely, normal density ρ_n and specific heat C_v in 3D resonance p -wave superfluids with A1-symmetry at low temperatures $T \ll T_C$. Our goal is to try to find nontrivial contributions to ρ_n and C_v from the nodal points on the mean-field level.

5.1 Specific heat in 3D A1-phase

The fermionic (quasiparticle) contribution to C_v on the mean-field level yields in 3D (see [48]):

$$C_v = \int \frac{\partial n_0(E_p/T)}{\partial T} E_p \frac{d^3 \vec{p}}{(2\pi)^3}, \quad (19)$$

where $n_0(\frac{E_p}{T}) = \frac{1}{(e^{\frac{E_p}{T}} + 1)}$ is quasiparticle distribution function, E_p is quasiparticle energy given by (5).

The calculations yield an expected result:

$$C_v \sim N(0) \frac{T^3}{\Delta_0^2} \quad (20)$$

deep in the BCS-domain, where $N(0) = \frac{mp_F}{2\pi^2}$ is the density of states at the Fermi-surface. Deep in the BEC-domain C_v is exponential:

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi^2} \frac{E_b}{4T^2} e^{-\frac{|E_b|}{2T}}, \quad (21)$$

where $|E_b|$ is given by (11).

Finally in the interesting region of small μ and low temperatures ($|\mu| \ll T \ll \Delta_0^2/\varepsilon_F$ and thus for the classical limit $\frac{|\mu|}{T} \rightarrow 0$) we have a nontrivial temperature dependence for C_v :

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi^2} \frac{\varepsilon_F T}{\Delta_0^2}. \quad (22)$$

Note that in the opposite quantum limit $\frac{T}{|\mu|} \rightarrow 0$ ($T \ll |\mu| \ll \frac{\Delta_0^2}{\varepsilon_F}$) we have:

$$C_v \sim \frac{1}{2\pi^2} \frac{\varepsilon_F T}{\Delta_0^2} \frac{T^2 m^{3/2}}{\mu^{1/2}} \quad (23)$$

in BCS-domain and respectively:

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi^2} \frac{|\mu|^3}{T^3} e^{-\frac{|\mu|}{T}} \frac{\varepsilon_F T}{\Delta_0^2} \quad (24)$$

in BEC-domain. In this limit C_v behaves very differently in BCS and BEC-domains.

For $|\mu| \sim T$ these results (23), (24) coincide by the order of magnitude with (22).

For small $|\mu|$ but intermediate temperatures $|\mu| \ll \Delta_0^2/\varepsilon_F \ll T \ll \Delta_0$ we recover a more expected result:

$$C_v \sim \frac{(2mT)^{3/2}}{2\pi^2}. \quad (25)$$

However at these temperatures bosonic contribution (contribution from the sound waves) prevails and yields:

$$C_v^B \sim \frac{T^3}{c_s^3} \frac{1}{2\pi^2}, \quad (26)$$

where the sound velocity c_s is given by (9) in BCS-domain and by (13), (17) in BEC-domain and close to $\mu = 0$.

We can see that it is possible to separate a power-law fermionic contribution $C_v \sim T^{5/2}$ at low temperatures and $C_v \sim T^{3/2}$ at intermediate temperatures from bosonic one $C_v^B \sim T^3$ close to the important point $\mu = 0$. We also see very different behavior of C_v in BCS and BEC-domains in the limit $\frac{T}{|\mu|} \rightarrow 0$.

Analogously in 2D 100%-polarized $(p_x + ip_y)$ -phase in quantum limit $T \ll |\mu| \ll \frac{\Delta_0^2}{\varepsilon_F}$ ($\frac{T}{|\mu|} \rightarrow 0$) we have for a quasiparticle contribution:

$$C_v \sim \frac{1}{2\pi} \frac{m\varepsilon_F}{\Delta_0^2} T^2 \quad (27)$$

in BCS-domain for $\mu \rightarrow +0$. Note that phonon contribution has the same order of magnitude as a fermionic contribution in BCS-domain.

In BEC-domain for $\mu \rightarrow -0$:

$$C_v \sim \frac{1}{2\pi} \frac{m\varepsilon_F}{\Delta_0^2} \frac{|\mu|^3}{T} e^{-\frac{|\mu|}{T}}. \quad (28)$$

Note that both in 3D and 2D for $T \neq 0$ we are effectively always in a classical limit $\frac{|\mu|}{T} \rightarrow 0$ since μ is continuous close to $\mu = 0$. Hence the real phase transition takes place only at $T = 0$ [37–41].

5.2 Normal Density in 3D A1-Phase

The quasiparticle contribution to the normal density in 3D A1-phase yields (see [48]):

$$\rho_n = -\frac{1}{3} \int p^2 \frac{\partial n_0(E_p/T)}{\partial E_p} \frac{d^3 \vec{p}}{(2\pi)^3}. \quad (29)$$

Deep in the BCS-domain the evaluation of ρ_n yields:

$$\rho_n \sim \rho \frac{T^2}{\Delta_0^2} \quad (30)$$

where $\rho = mn$ is a total mass-density. Note that rigorously speaking equation (30) yields longitudinal component of normal density tensor ρ_{nl} . There is also a small transverse contribution $\rho_{nt} \sim T^4$ firstly obtained by [37–40].

Deep in the BEC-domain the normal density is exponential and reads:

$$\rho_n \sim \frac{m}{\pi^2} (2mT)^{3/2} e^{-\frac{E_b}{2T}}. \quad (31)$$

Finally close to $\mu = 0$ at low temperatures ($|\mu| \ll T \ll \Delta_0^2/\varepsilon_F$ and thus in the classical limit $\frac{|\mu|}{T} \rightarrow 0$) we have:

$$\rho_n \sim \frac{m}{\pi^2} (2mT)^{3/2} \frac{\varepsilon_F T}{\Delta_0^2}. \quad (32)$$

In the opposite quantum limit $\frac{T}{|\mu|} \rightarrow 0$ ($T < |\mu| < \frac{\Delta_0^2}{\varepsilon_F}$) we have:

$$\rho_n \sim \frac{m}{\pi^2} \frac{\varepsilon_F T}{\Delta_0^2} 2mT (2m|\mu|)^{1/2} \quad (33)$$

in BCS-domain and respectively:

$$\rho_n \sim \frac{m}{\pi^2} \frac{\varepsilon_F T}{\Delta_0^2} e^{-\frac{|\mu|}{T}} 2m|\mu|(2mT)^{1/2} \quad (34)$$

in BEC-domain, so again the behavior of ρ_n is very different in BCS and BEC-domains in the quantum limit.

For $|\mu| \sim T$ these results (33), (34) coincide by the order of magnitude with (32).

At intermediate temperatures $|\mu| \ll \Delta_0^2/\varepsilon_F \ll T \ll \Delta_0$ a normal density yields as expected:

$$\rho_n \sim \frac{m}{\pi^2} (2mT)^{3/2}. \quad (35)$$

However at these temperatures bosonic (phonon) contribution from the sound waves prevails and yields (see [48]):

$$\rho_n^B \sim \frac{T^4}{c_s^5}, \quad (36)$$

where c_s is again given by (5), (13) and (17) in BCS-, BEC-domain and close to $\mu = 0$ respectively. We can again separate a fermionic (quasiparticle) contribution to ρ_n ($\rho_n \sim T^{5/2}$ at low temperatures and $\rho_n \sim T^{3/2}$ at intermediate temperatures) from bosonic one ($\rho_n \sim T^4$) close to the point $\mu = 0$. We also see a very different behavior of ρ_n in BEC and BCS-domains in the quantum limit $\frac{T}{|\mu|} \rightarrow 0$. The same behavior holds for a 2D case.

6 Orbital Waves, Intrinsic Angular Momentum and the Chiral Anomaly in A1-Phase

The topological effects in A1-phase are really pronounced in the spectrum of orbital waves and in the superfluid hydrodynamics at low temperatures $T \rightarrow 0$ especially in BCS-domain. Here by symmetry requirements we can write the following expression for the total mass-current:

$$\vec{j}_{tot} = \vec{j}_B + \vec{j}_{an}, \quad (37)$$

where

$$\vec{j}_{an} = -\frac{\hbar}{4m} C_0(\vec{l}, \text{rot}\vec{l})\vec{l} \quad (38)$$

is an anomalous current. In BEC-domain $C_0 = 0$ and anomalous current is absent. The reason for that is connected with the fact that the integral $\frac{N(0)}{2} \int d\xi_p (1 - \frac{\xi_p}{|\xi_p|}) = 0$ in BEC-domain (for $\xi_p > 0$), while it is nonzero and defines total density in BCS-domain. However it is a difficult question whether $C_0 = 0$ or not in the BCS-domain.

In the same time \vec{j}_B in (37) is a total mass-current in BEC-domain for p -wave molecules. It reads:

$$\vec{j}_B = \rho \vec{v}_S + \frac{\hbar}{2m} \text{rot} \frac{\rho \vec{l}}{2}, \quad (39)$$

where $\vec{L} = \frac{\hbar\rho}{2m}\vec{l}$ is the density of orbital momentum, \vec{v}_S is a superfluid velocity.

Anomalous current \vec{j}_{an} violates conservation law for total mass-current (total linear momentum) \vec{j}_{tot} since it cannot be expressed as a divergence of some momentum tensor Π_{ik} :

$$\frac{\partial j_{tot}^i}{\partial t} \neq -\frac{\partial}{\partial x_k}(\Pi_{ik}). \quad (40)$$

Thus the presence of anomalous current destroys the superfluid hydrodynamics of the A1-phase at $T \rightarrow 0$. Its contribution to the equation for total linear momentum (to $\frac{\partial j_{tot}^i}{\partial t}$) can be compensated only by adding the term with a relative normal velocity and normal density ρ_n ($T = 0$)($\vec{v}_n - \vec{v}_S$) to the total current \vec{j}_{tot} already at $T = 0$ (see [8–12]). The anomalous current also changes significantly the spectrum of orbital waves. This additional Goldstone branch of collective excitations in the A1-phase is connected with the rotation of \vec{l} -vector around perpendicular axis. It is quadratic at low frequencies (A1-phase is called an orbital ferromagnet. It is also spin ferromagnet). However, the coefficient in front of q^2 is drastically different in BCS- and BEC-domains.

In BEC-domain for small ω and \vec{q} :

$$\omega \sim \frac{q^2}{m}. \quad (41)$$

At the same time in BCS-domain:

$$\frac{(\rho - C_0)}{\rho}\omega \sim \frac{q^2}{m}. \quad (42)$$

The most straightforward way to get (41) and (42) is to use diagrammatic technique of [49, 50] for collective excitations spectrum in p -wave and d -wave superfluids. If according to [20] the relative difference $\frac{(\rho - C_0)}{\rho} \sim \frac{\Delta_0^2}{\varepsilon_F^2} \ll 1$ then the coefficient in front of ω in (42) is much smaller in BCS-domain than in BEC-domain (for instance in superfluid $^3\text{He-A}$: $\frac{\Delta_0}{\varepsilon_F} \sim \frac{T_C}{\varepsilon_F} \sim 10^{-3}$ and thus $\frac{(\rho - C_0)}{\rho} \sim 10^{-6}$).

The same estimate for the density of intrinsic angular momentum yields $\vec{L} = \frac{\hbar}{2m}(\rho - C_0)\vec{l}$. Note that even in BCS-case different calculations yield here different results. For $\vec{l} = \text{const}$ the evaluations of [51–57] yield $\vec{L} = \frac{\hbar}{2m}\rho\vec{l}$ while the inclusion of the inhomogeneous textures of \vec{l} -vector restores $\vec{L} = \frac{\hbar}{2m}(\rho - C_0)\vec{l}$.

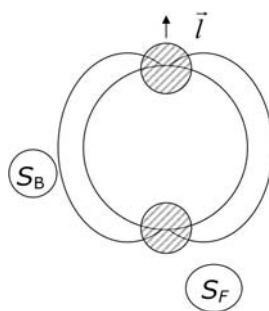
Note that accordingly to Leggett [58] the total N -particles Hamiltonian \hat{H} exactly commutes with the z -projection of the angular momentum $\hat{L}_z = \frac{\hbar}{2}\hat{N}$. This fact is in favor of the result $\vec{L} = \frac{\hbar}{2m}\rho\vec{l}$ for $\vec{l} = \text{const}$.

Returning back to the complicated problem of chiral anomaly let us recollect the two different approaches to this problem worked out in the end of eighties.

7 Supersymmetric Hydrodynamics of the A1-Phase

The first approach [7] is based on supersymmetric hydrodynamics of the A1-phase.

Fig. 4 The qualitative illustration of fermionic (S_F) and bosonic (S_B) contributions to the total hydrodynamic action S_{tot} of the A1-phase at $T \rightarrow 0$



The idea of [7] was to check whether the chiral anomaly (or more precisely a term $\vec{j}_{an}\vec{v}_S$ in total energy) is directly connected with the zeroes of the gap. The authors of [7] assumed that in 3D condensed matter system at low frequencies the only physical reason for anomaly can be connected with the infrared singularity. Note that ultraviolet singularities are absent in condensed matter systems in contrast with quantum electrodynamics. Strong (critical) fluctuations are also suppressed in 3D system. Thus the main idea of [7] was to check the dangerous infrared regions where the gap is practically zero. To do that the authors of [7] consider the total hydrodynamic action S_{tot} of the A1-phase for low frequencies and small q -vectors as a sum of bosonic and fermionic contributions:

$$S_{tot} = S_B + S_F, \quad (43)$$

where $S_B(\rho, \vec{l}, \vec{v}_S)$ is a bosonic action and S_F is a fermionic action connected with the zeroes of a superfluid gap (see Fig. 4).

Generally speaking the idea of [7] was to describe by the supersymmetric hydrodynamics all the zero energy Goldstone modes including the fermionic Goldstone mode which comes from the zero of the gap.

The authors of [7] were motivated by the nice paper [59] where for the first time a massless fermionic neutrino was included in the effective infrared Lagrangian for electro-weak interactions.

To be more specific bosonic action S_B in (43) describes sound waves and orbital waves in purely bosonic limit and does not contain an anomalous term $\vec{j}_{an}\vec{v}_S$ in the bosonic energy. Formally $S_B = \int L_B d^4x$, where $x = (\vec{r}, t)$ and bosonic Lagrangian $L_B = \vec{L}\dot{\vec{\theta}} - E_B$ where $\dot{\vec{\theta}}$ —is angular frequency. It corresponds to bosonic energy E_B which has the form:

$$E_B = E_{OB} + \frac{j_B^2}{2\rho}, \quad (44)$$

where E_{OB} is connected with thermodynamic and liquid crystal like orbital energy. Thus an action S_B describes 3 equations for bosonic hydrodynamics at $T = 0$; namely the equation of mass-conservation:

$$\partial\rho/\partial t + \text{div } \vec{j}_B = 0, \quad (45)$$

the equation for \vec{l} -vector dynamics:

$$\dot{\vec{l}} + (\vec{v}_S \vec{\nabla}) \vec{l} + \frac{2m}{\hbar \rho} \left[\vec{l}, \frac{\partial E_{OB}}{\partial \vec{l}} - \partial_i \frac{\partial E_{OB}}{\partial_i \vec{l}} \right] = 0, \quad (46)$$

and the equation for superfluid velocity dynamics:

$$\dot{\vec{v}}_S + \vec{\nabla} \left(\mu + \frac{v_S^2}{2} + \frac{\hbar}{4m} (\vec{l} \text{ rot } \vec{v}_S) \right) = \frac{\hbar}{2m} \vec{\nabla} l_p [\vec{l} \dot{\vec{l}}]_p. \quad (47)$$

The right-hand side of the last equation expresses the famous Mermin-Ho identity for $^3\text{He-A}$ [20]. The system of these 3 equations are in agreement with the equations of mass-current conservation $\frac{\partial j_i^B}{\partial t} + \frac{\partial}{\partial x} \Pi_{in} = 0$ and the equation of total energy conservation $\frac{\partial E_B}{\partial t} + \text{div } \vec{Q} = 0$. They contain sound wave $\omega = c_S k$ and orbital wave $\omega \sim k^2/m$.

Fermionic part of the action in (43) describes fermionic Goldstone mode and has the form:

$$S_F = \int d^4x \Psi^* \hat{\Lambda} \Psi, \quad (48)$$

where doublets Ψ^* and Ψ are fermionic (grassman) variables and

$$\begin{aligned} \hat{\Lambda} = & i \frac{\partial}{\partial t} + \begin{pmatrix} i v_l \vec{l} (\vec{\nabla} - i p_F \vec{l}) & i v_l \vec{\Delta} \vec{\nabla} \\ i v_l \vec{\Delta}^* \vec{\nabla} & -i v_l \vec{l} (\vec{\nabla} + i p_F \vec{l}) \end{pmatrix} \\ & + \begin{pmatrix} i \frac{v_l}{2} \text{div } \vec{l} & i \frac{v_l}{2} \text{div } \vec{\Delta} \\ i \frac{v_l}{2} \text{div } \vec{\Delta}^* & -i \frac{v_l}{2} \text{div } \vec{l} \end{pmatrix} + a(\vec{l} \text{ rot } \vec{l}) + b(\vec{v}_S \vec{l}), \end{aligned} \quad (49)$$

where $\hat{\Lambda}$ is Lagrangian operator for nodal fermions. Here v_t, v_L, a, b —are phenomenological coefficients. This Lagrangian is valid both in weak and strong-coupling limits. In weak-coupling case $v_t \sim \Delta_0/\varepsilon_F v_F, v_L \sim v_F$ —for “transverse” and “longitudinal” Fermi-velocities, $\vec{\Delta} = (\vec{e}_1 + i \vec{e}_2) \Delta_0$ and $\vec{v}_S = \hbar/2me_1 \vec{\nabla} e_{2i}$. The long derivatives $(\vec{\nabla} \pm i p_F \vec{l})$ describe fermionic Goldstone variables with practically zero energy near north $\Psi_1(\vec{p} \approx p_F \vec{l})$ and south $\Psi_2(\vec{p} \approx -p_F \vec{l})$ poles. Remind that the quasiparticle energy: $E_p = \sqrt{\xi^2(p) + \frac{\Delta_0^2 |\vec{l} \vec{p}|^2}{p_F^2}} = 0$ for $\vec{p} = \pm p_F \vec{l}$. Accordingly the doublet for Ψ has the form $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$.

To come from our supersymmetric action in (43) to a standard action which contains only bosonic fields we must eliminate fermionic degrees of freedom in (39) by calculating the functional integral over Grassman variables. This yields:

$$S_{eff} = S_B + \Delta S_B, \quad (50)$$

where $\Delta S_B = \ln \int D\Psi D\Psi^* \exp(\int d^4x \Psi^* \hat{\Lambda} \Psi) = \ln \text{Det}(\hat{\Lambda} \hat{\Lambda}_0^{-1}) = \text{Tr} \ln(\hat{\Lambda} \hat{\Lambda}_0^{-1})$, and $\hat{\Lambda}_0$ is normalizing operator.

In hydrodynamic ideology it coincides with $\hat{\Lambda}$ in homogeneous case, where $\vec{l} = \text{const}$ and $\vec{\Delta} = \Delta_0(\vec{e}_1 + i \vec{e}_2) = \text{const}$.

If we hope to obtain an anomalous term in current $\vec{j}_{an} = -\frac{\hbar}{4m}C_0(\vec{l}\text{rot}\vec{l})\vec{l}$ with large coefficient $C_0 \sim \rho$ we must get the term $\vec{j}_{an}\vec{v}_S$ in ΔL_B ($\Delta S_B = \int \Delta L_B d^4x$). But due to small fermionic density $\rho_F \sim \Psi^*\Psi$ near the south and north poles (small statistical weight of fermionic pockets on Fig. 4 in comparison with total density ρ) we have to get $C_0 \sim \delta(\vec{p} \pm p_F\vec{l})$ in momentum space or accordingly to $C_0 = \text{const}$ in real space. That is why in order to obtain an anomalous term in current we must find very strong delta-functional infra-red divergencies in ΔS_B .

But a very accurate calculation of the functional integral (of $\text{Tr} \ln \hat{\Lambda} \hat{\Lambda}_0^{-1}$) in hydrodynamic limit ($\omega \rightarrow 0, k \rightarrow 0$ or accordingly \vec{r} and $t \rightarrow \infty$) will give us a final result:

$$\Delta L_B = \frac{p_F^2 v_L}{32\pi^2} \left\{ [\vec{l}\text{rot}\vec{l}]^2 + \frac{v_L^2}{v_l^2} (\vec{l}\text{rot}\vec{l})^2 \right\} \ln \frac{x_{\max}}{x}, \quad (51)$$

where $x_{\max} \sim l_{MF}$ in hydrodynamic regime at $T = 0$, l_{MF} —is a mean-free path, $\xi_0 < x < l_{MF}$, $\xi_0 \sim \frac{v_F}{\Delta_0}$ is a coherence length.

It means that only weak logarithmic type of infra-red singularities is present in the system and we do not have in it a strong δ -functional singularity.

These weakly (logarithmically) divergent terms lead only to a strong renormalization of liquid crystal type coefficients K_2 and K_3 in E_{OB} [60, 61] (and after Fourier transform in the spectrum of orbital waves $\omega \sim \frac{q^2}{m} \ln \frac{\Delta_0}{v_F q}$) but they do not lead to the appearance of large anomalous coefficient C_0 and thus do not destroy the hydrodynamics. Hence even if chiral anomaly exists in the BCS-domain of 3D A1-phase, it is not directly connected with the dangerous regions of momentum space near zeroes of the gap (it does not have an infra-red character).

8 The Different Approach Based on the Formal Analogy with Quantum Electrodynamics

The authors of [8–12] have the different also rather nice approach based on the formal analogy between the anomalous current in 3D $^3\text{He-A}$ and chiral anomaly in QED-theory. They assume that anomalous current with the coefficient $C_0 \sim \rho$ in BCS-domain of the A1-phase is not directly connected with the zeroes of the gap (thus it is not contained even in the supersymmetric hydrodynamics). They think that it is connected with the global *topological* considerations and thus should be added to (43) by hands. To illustrate this point they solve microscopic Bogolubov-de-Gennes (BdG) equations for fermionic quasiparticles in a given twisted texture ($\vec{l} \parallel \text{rot}\vec{l}$) of the \vec{l} -vector. To be more specific they consider the case:

$$\vec{l} = \vec{l}_0 + \partial\vec{l} \quad (52)$$

where

$$l_z = l_{0z} = e_z; \quad l_y = \delta l_y = Bx; \quad l_x = 0. \quad (53)$$

In this case:

$$\vec{l}\text{rot}\vec{l} = l_z \frac{\partial l_y}{\partial x} = B = \text{const} \quad (54)$$

and accordingly

$$\vec{j}_{an} = -\frac{\hbar}{4m} C_0 B \vec{e}_z. \quad (55)$$

After linearization BdG equations become equivalent to Dirac equation in homogeneous magnetic field $B = (\vec{l} \text{rot} \vec{l})$.

Namely after linearization BdG equation reads $\hat{H}\chi = E\chi$, where the doublet χ has the form $\chi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \exp(ip_z z + ip_y y)$ and for the Hamiltonian we have:

$$\hat{H} = \xi(p_z) \hat{\sigma}_3 + v_\perp \left(\hat{\sigma}_1 \frac{1}{i} \partial_x - \hat{\sigma}_2 (p_y - eBx) \right), \quad (56)$$

where $\{\hat{\sigma}_3, \hat{\sigma}_2, \hat{\sigma}_1\}$ —are Pauli matrices, $\xi(p_z) = \frac{p_z^2 - p_F^2}{2m}$, $v_t = v_F \frac{\Delta_0}{\varepsilon_F}$ —corresponds to weak-coupling limit, and $e = \frac{p_z}{p_F} = \pm 1$ —is an electric charge.

The solution of BdG equations yields for the doublets $\chi(x)$:

$$\chi_{n_L}(x) = \theta(-eB) \begin{pmatrix} \alpha_{n_L} f_{n_L}(\tilde{x}) \\ i\beta_{n_L} f_{n_L-1}(\tilde{x}) \end{pmatrix} + \theta(eB) \begin{pmatrix} \alpha_{n_L} f_{n_L-1}(\tilde{x}) \\ i\beta_{n_L} f_{n_L}(\tilde{x}) \end{pmatrix}, \quad (57)$$

where θ —is a step-function, $f_{n_L}(\tilde{x}) = f_{n_L}(x - \frac{p_y}{eB})$ —is the ortho-normalized wave-function of harmonic oscillator, $f_{-1} = 0$, $|\alpha_{n_L}|^2 = \frac{E_{n_L} + \xi(p_z)}{2E_{n_L}}$, $|\beta_{n_L}|^2 = \frac{E_{n_L} - \xi(p_z)}{2E_{n_L}}$ —are Bogolubov coefficients ($|\alpha_{n_L}|^2 + |\beta_{n_L}|^2 = 1$).

Accordingly for the spectrum:

$$E_{n_L}(p_z) = \pm \sqrt{\xi^2(p_z) + \tilde{\Delta}_{n_L}^2}, \quad (58)$$

where $\tilde{\Delta}_{n_L}^2 = 2n_L v_t^2 p_F |eB|$ is a gap squared and n_L is a quantum number for Landau level. The solution for χ_{n_L} and E_{n_L} contains the level asymmetry.

Namely for $n_L \neq 0$ (see Fig. 5) all the levels are gapped $\tilde{\Delta}_{n_L} \neq 0$ and doubly degenerate with respect to $p_z \rightarrow -p_z$. Their contribution to total mass-current is zero for $T \rightarrow 0$.

However for $n_L = 0$ there is no gap $\tilde{\Delta}_0 = 0$ and we have an asymmetric chiral branch which exists only for $p_z < 0$ (only for one sign of eB). The energy spectrum for $n_L = 0$ yields:

$$E_0 = \xi(p_z). \quad (59)$$

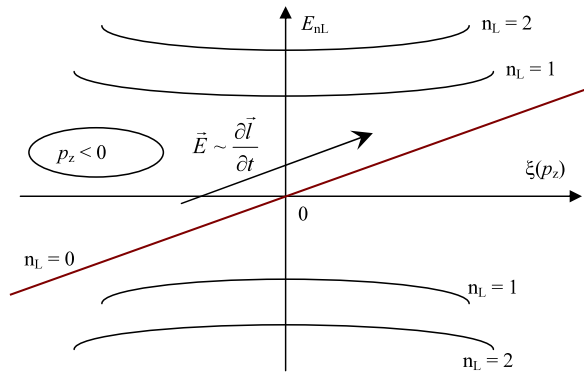
We can say that there is no gap for zeroth Landau level. Moreover in BCS-domain $E_0 = 0$ for $|p_z| = p_F$ —the chiral level crosses the origin in Fig. 5, so we have a zero mode.

Note that in BEC-domain $E_0 \geq |\mu|$ and the zeroth Landau level does not cross the origin. The absence of a zero-mode in BEC-domain is a physical reason why $C_0 = 0$ here.

The zeroth Landau level gives an anomalous contribution to the total current in BCS-domain:

$$\vec{j}_{an}(\vec{r} = 0) = -\vec{e}_z (\vec{l} \text{rot} \vec{l}) \int_{p_z < 0} \frac{p_z}{2\pi^2} d\xi(p_z) = -\frac{\hbar C_0}{4m} (\vec{l} \text{rot} \vec{l}) \vec{l}, \quad (60)$$

Fig. 5 (Color online) The level structure of the Dirac equation in magnetic field $B = (\vec{l} \text{ rot } \vec{l})$. All the levels with $n_L \neq 0$ are doubly degenerate. The zeroth level is chiral. It crosses the origin for $|p_z| = p_F$ in BCS-domain ($\mu > 0$). We also illustrate the concept of the spectral flow which will be discussed in Sect. 9



where

$$\frac{(\vec{l} \text{ rot } \vec{l}) p_z}{2\pi^2 p_F} = \frac{eB}{2\pi^2} = \int |f_0|^2 \frac{dp_y}{2\pi} \quad (61)$$

and as a result:

$$C_0 \approx m \frac{p_F^3}{6\pi^2} \approx \rho \quad (62)$$

in BCS-domain.

It is easy to observe that the integral for C_0 in (60), (61) is governed by the narrow cylindrical tube inside the Fermi-sphere (see Fig. 6) with the length p_F parallel to the \vec{l} -vector and radius of the cylinder squared given by:

$$\langle p_y^2 \rangle \sim p_F |eB|. \quad (63)$$

According to the ideas of [8–10, 37–40] this tube plays the role of a vortex in momentum space, thus providing a normal core and anomalous current at $T = 0$.

Note that a key result of [8–12] connected with the absence of the gap for the energy of zeroth Landau level (see (59)) is pretty stable with respect to small modifications of the texture of the \vec{l} -vector in (53). The careful analysis fulfilled by the authors of present paper shows that an account of small bending corrections with $[\vec{l} \text{ rot } \vec{l}] \neq 0$ (small tilting of magnetic field with respect to (x, y) plane $\vec{B} = B_0 \vec{e}_z + B_1 \vec{e}_x$) as well as small inhomogeneties of a magnetic field $B = B_0 + B_1 x$, which lead to the double-well effective potential, do not suppress a zero-mode in the spectrum of BdG-equation (do not lead to the appearance of a gap $\tilde{\Delta}_{n=0}$ for zeroth Landau level).

9 How to Reach Hydrodynamic Regime $\omega\tau \ll 1$

Inspite of the zero-mode stability the authors of [7] expressed their doubts with respect to the calculation of C_0 based on Dirac equation in homogeneous magnetic field $B = (\vec{l} \text{ rot } \vec{l})$. From their point of view the calculation of C_0 from (60), (61) is an oversimplification of a difficult many-particle problem. In particular they emphasized the role of the finite damping $\gamma = 1/\tau$ as well as other residual interactions

Fig. 6 The contribution to the coefficient C_0 is governed by the narrow cylindrical tube of the length p_F and the width $\langle p_y^2 \rangle \sim p_F |eB|$ inside the Fermi-sphere

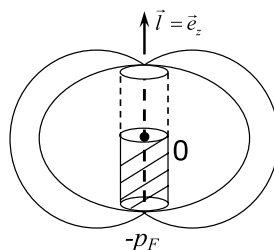


Fig. 7 The possible role of the damping in reaching hydrodynamic limit for low frequencies and small \vec{k} -vectors when $\gamma > \omega_0$ ($\omega_0 = E_1 - E_0$ —is level spacing)

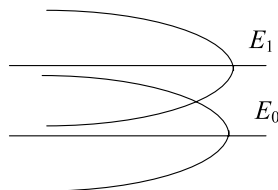
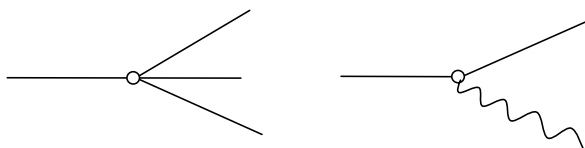


Fig. 8 Different decay processes for damping of chiral fermions at $T = 0$: the standard three-fermion decay process and a decay process with the emission of orbital wave



to destroy the chiral anomaly, which is connected with the states inside the Fermi sphere, thus restoring the superfluid hydrodynamics (without normal velocity \vec{v}_n and normal density ρ_n). Indeed, if the damping γ is larger than the level spacing of the Dirac equation:

$$\omega_0 = v_t p_F \sqrt{\frac{|\vec{l} \text{rot} \vec{l}|}{p_F}} \quad (64)$$

in case when $\xi(p_z) = 0$, then the contribution from the zeroth Landau level should be washed out by damping (see Fig. 7) and we will reach hydrodynamic regime. The damping γ for chiral fermions (for fermions living close to the nodes) in a very clean A1-phase without impurities is defined at $T = 0$ by the different decay processes (see [48]).

It is natural to assume that the only parameter which defines γ at $T = 0$ for chiral fermions is $\Delta_0 \langle \theta \rangle = \Delta_0 \frac{\langle p_{\perp} \rangle}{p_F}$. The leading term in decay processes is given by the emission of an orbital wave (see Fig. 8). It reads

$$\gamma \propto \left[\frac{\Delta_0^2 p_{\perp}^2 / p_F^2 + v_F^2 (p_z - p_F)^2}{\varepsilon_F} \right]. \quad (65)$$

For $p_z = p_F$ ($\xi(p_z) = 0$):

$$\gamma \sim \frac{\Delta_0^2}{\varepsilon_F} \frac{p_{\perp}^2}{p_F^2}. \quad (66)$$

Note that for chiral fermions on zeroth Landau level

$$\frac{\langle p_{\perp} \rangle}{p_F} = \left(\frac{|\vec{l} \text{rot} \vec{l}|}{p_F} \right)^{1/2} \quad (67)$$

and the level spacing for $\xi(p_z) = 0$ reads:

$$\omega_0 \sim \Delta_0 \frac{\langle p_{\perp} \rangle}{p_F}. \quad (68)$$

Hence $\gamma/\omega_0 \ll 1$ close to zero mode for these two decay processes and we are in a ballistic regime. Thus it is difficult to wash out the contribution from the zeroth Landau level by the different decay processes in superclean $^3\text{He-A1}$ -phase at $T = 0$. Note that a hydrodynamic regime $\omega\tau \ll 1$ could be easily reached in the presence of non-magnetic impurities or in the presence of aerogel [62–65].

9.1 The Concept of Spectral Flow and Exact Anomalies Cancellation

If anomalous current exists in superclean A1-phase at $T = 0$ it should be compensated somehow. According to [8–10] the deficit in the equation for the conservation of total linear momentum due to the presence of anomalous current:

$$\frac{\partial j_{an}^i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = I, \quad (69)$$

where $I = \frac{3\hbar}{4m} C_0 \vec{l}(\text{rot} \vec{l} \frac{\partial \vec{l}}{\partial t})$, is exactly compensated by quasiparticle contribution \vec{P}_{quas} :

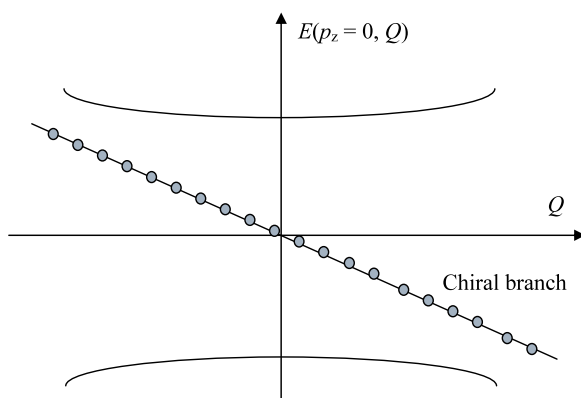
$$\frac{\partial P_{quas}^i}{\partial t} + \frac{\partial \Phi_{ik}}{\partial x_k} = -I, \quad (70)$$

where $\vec{P}_{quas} = \rho_n(T=0)(\vec{v}_n - \vec{v}_S)$ in hydrodynamic regime.

Note that $\rho_n(T=0) \sim \frac{|\vec{l} \text{rot} \vec{l}|}{\Delta_0}$ is non-analytical and connected with non-zero bending. The arguments of [8–10] are connected with the non-conservation of axial current j_5 in QED which just compensates I via a Schwinger term $\vec{E} \vec{B} \sim \frac{\partial \vec{l}}{\partial t} \text{rot} \vec{l}$. Physically, according to [8–10, 66, 67], this cancellation is due to the spectral flow from negative to positive energy values along the anomalous branch with $n_L = 0$ on Fig. 5 and then to the quasiparticle bath in the presence of an electric field $\vec{E} \sim \frac{\partial \vec{l}}{\partial t}$ (of a time-dependent texture of the \vec{l} -vector).

Note that in the physics of a vortex-core in case of cylindrical symmetry there is one anomalous level which crosses the zero energy (see Fig. 9). As a function of generalized angular momentum Q it represents at $T = 0$ the set of discrete points separated by a minigap $\omega_0 \sim \Delta_0^2/\varepsilon_F$. Thus at $T = 0$ and in a superclean case $\gamma = 1/\tau \rightarrow 0$ the spectral flow from negative to positive energies is totally suppressed. Thus the Thouless result [68] for the Berry-phase without anomaly is restored for the physics of vortex friction. An inclusion of large number of impurities or finite temperature leads to the revival of anomaly in hydrodynamic regime $\omega_0\tau \ll 1$ in

Fig. 9 (Color online) Levels structure in the vortex core of $^3\text{He-A}$. All the branches are even in generalized angular momentum Q , but one branch $E(p_z = 0, Q) = -\omega_0 Q$ which crosses zero energy at $Q = 0$ is chiral (odd in Q). It participates in momentum exchange between fermions in the vortex core and fermions of the heat bath in the hydrodynamic limit $\omega_0 \tau \ll 1$ accordingly to [37–40]



case of vortices. So, we could assume that the chiral anomaly and the spectral flow are ineffective at $T = 0$ both for vortices and for a bulk A-phase of superfluid ^3He in a superclean limit. Thus the question how the total linear momentum is conserved in this case in an infinite system (without the walls) is very untrivial and unresolved yet.

We think that exact cancellation between time-derivatives of anomalous and quasiparticle currents should be demonstrated explicitly by the derivation and solution of kinetic equations for nodal quasiparticles both in ballistic and hydrodynamic regimes. Note that if $T \neq 0$ (as we always have in real experiment) and for low frequencies $\omega \tau (T \neq 0) \ll 1$ ($\tau (T \neq 0) \sim \alpha T^{-n}$) a relative normal velocity $\vec{v}_n - \vec{v}_S = \frac{\partial \varepsilon_0}{\partial \vec{p}_{quas}}$ becomes an additional hydrodynamic variable and so the cancellation of the linear momentum deficit will be fulfilled automatically.

Thus the question with exact anomalies compensation exists only for $T = 0$. Note that an approach based on kinetic equation for quasiparticles at different temperatures and impurities concentrations in a vortex core of s -wave superconductors and superfluid ^3He was worked out by [69–72] for the case of a singular vortex.

In case of nonsingular vortex structures in $^3\text{He-A}$ we should mention here also the articles of [66, 67] where in finite systems the author consider the scattering of quasiparticles on the walls of the container to get a finite γ at $T = 0$. The importance of prehistory of orbital texture in the framework of the concept of a spectral flow was also stressed in these papers.

10 Conclusions

We solve the Leggett equations and construct the phase-diagram of BCS-BEC crossover at low temperatures $T \ll T_C$ for 100%-polarized 3D A1-phase. From the evaluation of low temperature specific heat and normal density we see the indications of quantum phase—transition close to the point $\mu(T = 0) = 0$. In the same time deep in BCS and BEC-domains the crossover ideas of Leggett and Nozieres, Schmitt-Rink work pretty well. In these regions the phase-diagram for p -wave gas resembles in gross features the s -wave case. We discuss a complicated problem of chiral anomaly and mass-current non-conservation in BCS A1-phase at $T = 0$. We present two

different approaches to this problem—one based on supersymmetric hydrodynamics, another one—on the formal analogy with Dirac equation in QED-theory. We evaluate the damping $\gamma = 1/\tau$ due to different decay processes in superclean BCS A1-phase at $T = 0$ and find that γ is small in comparison with level spacing ω_0 of BdG-equation. To reach the hydrodynamic regime $\omega\tau \ll 1$ we need a sufficient amount of aerogel or nonmagnetic impurities at $T = 0$. We assume that both in hydrodynamic and in a ballistic regimes at $T = 0$ we have to derive a reliable kinetic equation to demonstrate explicitly an exact cancellation between time-derivatives of anomalous current $\vec{j}_{an} = -\frac{\hbar}{4m}C_0\vec{l}(\vec{l}\text{rot}\vec{l})$ and quasiparticle contribution \vec{P}_{quas} in the equation for the conservation of the total linear momentum \vec{j}_{tot} . Note that for the full theoretical analysis of the problem the other residual interactions different from damping are also important for nodal fermions. To check whether a chiral anomaly has an infra-red manifestation (which was not caught in the approach based on supersymmetric hydrodynamics of [7]) it will be useful to derive a complete set of Ward identities between self-energies of chiral fermions Σ and the corresponding vertices Γ . The idea is to find in this approach either a strong infra-red singularity or a powerful reexpansion of quasiparticle spectrum for $\omega, \vec{k} \rightarrow 0$.

Note that the importance of residual Fermi-liquid like interactions for the analysis of half-integer vortex in 3D A-phase of ^3He was recently emphasized in [73].

We invite experimentalists to enter this very interesting problem. It will be interesting to measure a spectrum and damping of orbital waves in superfluid A-phase of ^3He at low temperatures $T \ll T_C$. The spectrum is quadratic for low frequencies $\omega < \Delta_0^2/\varepsilon_F$ and contains a density of intrinsic angular momentum as a coefficient near the linear in frequency term:

$$\frac{(\rho - C_0)}{\rho}\omega \sim \frac{k_Z^2}{m} \ln \frac{\Delta_0}{v_F k_Z}, \quad (71)$$

where $\vec{e}_z \parallel \vec{l}$.

At higher frequencies the spectrum is almost linear $\omega^2 \ln \frac{\Delta_0}{\omega} \sim v_F^2 k_Z^2 \ln \frac{\Delta_0}{v_F k_Z}$ [49, 50].

The damping of the orbital waves provides an evaluation of orbital viscosity in ^3He -A at low temperatures $T \ll T_C$. Note that even in this case it is an interacting possibility to get an overdamped (diffusive) character of the spectrum.

This possibility gains a theoretical support in papers of [74–76] where in the functional integral technique in the hydrodynamic limit of small ω and \vec{k} the author obtained several overdamped modes in partially polarized A1-phase.

Another possibility of an overdamped diffusive spectrum was considered by [77, 78] in the impurity diagrammatic technique [79, 80] for a hydrodynamic regime $\omega\tau \ll 1$ of spin waves in a frustrated 2D AFM. Note that in the opposite high-frequency regime the spectrum of spin waves is linear. Here it is possible to extend experiments of orbital inertia and orbital viscosity for nonsingular vortices in A-phase of ^3He on low temperatures $T \ll T_C$. Of course, to have A-phase at low temperatures we need a strong spin-polarization.

Note also that a crossover from the ballistic to the hydrodynamic regime $\omega\tau \ll 1$ could be fulfilled both due to aerogel (nonmagnetic impurities) or due to the final

temperature $T \neq 0$ which is always present in a real experiment. In the last case the damping $\gamma \sim T^n$ is temperature dependent.

Finally, to measure non-trivial topological effects in 2D we propose to perform the experiments with a Josephson current between two thin films or two magnetic taps: one with a 2D axial phase and topological charge $Q = 1$ [81] and another one with planar 2D-phase with $Q = 0$. In this type of experiments we hope it will be possible to measure directly $\Delta Q = 1$.

Another possibility is to measure the contribution of massless Majorana fermions for the edge states on the surface of superfluid $^3\text{He-B}$ and the rough wall (or on the surface of a vibrating wire in Lancaster experiments) [82, 83].

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